Lecture 10. Linear Second-Order Equations with Constant Coefficients Part 1

Linear Second-Order Equations with Constant Coefficients

Let's discuss how to solve the homogeneous second-order linear differential equation

$$ay'' + by' + cy = 0 \tag{1}$$

with constant coefficients a, b, and c.

Consider a function of the form $y = e^{rx}$. Observe that

$$y' = (e^{rx})' = re^{rx}, \qquad \text{and} \quad y'' = (e^{rx})'' = r^2 e^{rx}.$$

This suggest that we can try to find r such that when we substitute y, y' and y'' into Eq. (1), we will get zero on the left hand-side.

Example 1 Find the values of r such that $y(x) = e^{rx}$ is a solution of the given differential equation.

$$y'' + 2y' - 15y = 0$$

ANS: If $y(x) = e^{rx}$, then $y' = re^{rx}$, $y'' = r^2 e^{rx}$

So we need find r such that

$$r^2 \underline{e^{rx}} + 2r\underline{e^{rx}} - 15\underline{e^{rx}} = 0$$

$$\Rightarrow e^{rx}(r^2 + 2r - 15) = 0$$

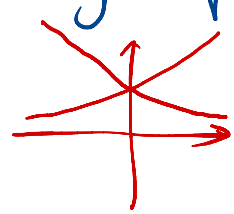
Note $e^{rx} \neq 0$ for any x

So $r^2 + 2r - 15 = 0$ (characteristic eqn)

$$\Rightarrow (r+5)(r-3) = 0 \Rightarrow r = -5 \text{ or } r = 3.$$

Thus $y_1 = e^{-5x}$ and $y_2 = e^{3x}$ are solutions to the given eqn.

Note y_1 and y_2 are linearly independent.



By Thm 4, $y(x) = c_1 y_1 + c_2 y_2 = c_1 e^{-5x} + c_2 e^{3x}$

is a general solution, where c_1 and c_2 are constants.

In general, we subsititute $y = e^{rx}$ in Eq. (1). Then

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Since e^{rx} is never zero. We conclude $y = e^{rx}$ will satisfy the differential equation in Eq. (4) precisely when r is a root of the algebraic equation

$$ar^2 + br + c = 0 \qquad r_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \tag{2}$$

This quadratic equation is called the **characteristic equation** of the homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

If Eq. (2) has distinct (unequal) roots r_1 and r_2, then the corresponding solutions $y_1(x) = e^{r_1x}$ and $y_2(x) = e^{r_2x}$ of Eq. (2). are linearly independent. Why?

Theorem 5 Distinct Real Roots

If the roots r_1 and r_2 of the characteristic equation in Eq. (2) are real and distinct, then

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

is a general solution of Eq. (1).

Question: What if we have $r_1 = r_2$ for the characeristic equation?

Example 2

Find general solutions of the given differential equations.

$$y'' + 4y' + 4y = 0 \qquad \circledast$$

ANS: The corresponding char. eqn is

$$r^2 + 4r + 4 = 0$$

$$\Rightarrow (r+2)^2 = 0 \Rightarrow r_1 = r_2 = -2$$

So $y_1 = e^{r_1x} = e^{r_2x} = e^{-2x}$ is a solution to $\circledast$.

How do we find another solution y_2 such that y_1 & y_2 are linearly independent?

Let's try $y_2 = xe^{-2x}$ $(= xy_1)$

$$y_2' = (xe^{-2x})' = x(e^{-2x})' + (x)'e^{-2x} = -2xe^{-2x} + e^{-2x}$$

$$y_2'' = -2e^{-2x} + (-2x)(-2)e^{-2x} - 2e^{-2x} = -4e^{-2x} + 4xe^{-2x}$$

Then LHS $= y_2'' + 4y_2' + 4y_2 = -4e^{-2x} + 4xe^{-2x} + 4(-2xe^{-2x} + e^{-2x}) + 4xe^{-2x}$

$= -\cancel{4e^{-2x}} + \cancel{4xe^{-2x}} - \cancel{8xe^{-2x}} + \cancel{4e^{-2x}} + \cancel{4xe^{-2x}} = 0 = RHS.$

So $y_2 = xe^{-2x}$ is a solution. And $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$ are linearly independent (check it using Wronskian)

By Thm 4. $y(x) = c_1 y_1 + c_2 y_2 \Rightarrow y(x) = (c_1 + c_2 x)e^{-2x}$ is a general solution.

In general, we have the following theorem if $r_1 = r_2$.

Theorem 6 Repeated Roots

If the characteristic equation in Eq. (2) has equal (necessarily real) roots $r_1 = r_2$, then,

$$y(x) = (c_1 + c_2 x)e^{r_1 x}$$

is a general solution of Eq. (2).

Exercise 3

Find general solutions of the given differential equations.

(1) $9y'' - 6y' + y = 0$

(2) $2y'' + 3y' = 0$

Solution.

(1) The corresponding characteristic equation is

$$\begin{aligned} & 9r^2 - 6r + 1 = 0 \\ \Rightarrow\quad & r^2 - \tfrac{2}{3}r + \tfrac{1}{9} = 0 \\ \Rightarrow\quad & \left(r - \tfrac{1}{3}\right)^2 = 0 \\ \Rightarrow\quad & r_1 = r_2 = \tfrac{1}{3} \end{aligned}$$

The general solution is $y = (c_1 + c_2 x)e^{\frac{1}{3}x}$, where c_1 and c_2 are constants,

(2) The corresponding characteristic equation is

$$\begin{aligned} & 2r^2 + 3r = 0 \\ \Rightarrow & r(2r + 3) = 0 \\ \Rightarrow & r = 0 \text{ or } r = -\frac{3}{2} \ (\text{distinct}) \end{aligned}$$

So $y = c_1 y_1 + c_2 y_2 = c_1 e^{0 \cdot x} + c_2 e^{-\frac{3}{2}x} = c_1 + c_2 e^{-\frac{3}{2}x}$ is a general solution.

Exercise 4.

Solve the initial-value problem $2y'' + 5y' - 3y = 0, y(0) = -5, y'(0) = 22$.

Solution.

The characteristic equation is given by:

$$2r^2 + 5r - 3 = 0$$

Solving this, we get

$$r_1 = \frac{1}{2} \text{ and } r_2 = -3$$

The general solution to the homogeneous differential equation is given by:

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{-3t}.$$

Now, we can use the initial conditions $y(0) = -5$ and $y'(0) = 22$ to find the values of c_1 and c_2.

As $y(0) = 5$, we have

$$y(0) = c_1 e^0 + c_2 e^0 = c_1 + c_2 = -5.$$

We compute $y'(t) = \frac{1}{2} c_1 e^{r_1 t} - 3c_2 e^{-3t}$

As $y'(0) = 22$, we have

$$y'(0) = \frac{1}{2} c_1 - 3c_2 = 22$$

Solving

$$\begin{aligned} c_1 + c_2 &= -5 \\ \frac{1}{2} c_1 - 3c_2 &= 22. \end{aligned}$$

We get $c_1 = 2$ and $c_2 = -7$.

Therefore

$$y(x) = 2e^{\frac{x}{2}} - 7e^{-3x}.$$